Control Design with Guaranteed Transient Performance: an Approach with Polyhedral Target Tubes

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Abstract—a novel approach is presented for control design with guaranteed transient performance for multiple-input multiple-output discrete-time linear polytopic difference inclusions. We establish a theorem that gives necessary and sufficient conditions for the state to evolve from one polyhedral subset of the state-space to another. Then we present an algorithm which constructs a time-varying output feedback law which guarantees that the state evolves within a time-varying polyhedral *target- tube* specifying the system's desired transient performance. We present generalisations involving constraints on the control, and a bounded additive disturbance term. Our formulation is very general and includes reference tracking with any desired transient behaviour in the face of disturbances, as specified, for example, by the most popular step response specifications. The approach is demonstrated by an example involving the control of water levels in two coupled tanks

1. Introduction

Background

- Rich control theories
- Few applications
- So for 30 years from 1987 control summit
- Math dominates control journals but poor math
- Difficult to get funding & jobs

Why so?

Negative:

- Control research in the literature has always focused on system stability, and related issues. Asymptotic analysis; who bother things of 10000Y?
- Another popular approach is optimization; and related issues. MPC, tuning facilities. “optimal” sounds great, but the objective makes sense?
- However, for real control applications, transient performance is far more important: a stable system with big transient errors or a very long settling time, is obviously unacceptable and cannot be used in practice by engineers.
- One could even say that only time-domain performance specifications matter for real control applications. This obvious gap exists for decades already and is the main cause why most modern control theories are not adopted in industry. There has been little research on control performance and those on it did not address explicit time-domain performance specifications such as rise-time, overshoot and settling-time, see the literature review section in this proposal.
- Classical designs such as pole placement and LQR fail to do so

Positive: Evidence of applicable control:

- Let us briefly review how the existing control designs handle performance as follows. The success that the proportional-integral-derivative (PID) controller has enjoyed in industry (it is often estimated that roughly 90% of industrial control systems use PID [1]) is not only because it is relatively simple to implement and easy to tune, but also because the design philosophy often addresses performance specifications, such as rise time, settling-time, over-shoot and steady-state error, though implicitly and mostly done in frequency domain.
- One of PID's shortcomings is the fact that its design does not explicitly take time domain performance requirements into account, and thus cannot ensure performance, resulting in engineers often resorting to their experience as well as trial-and-error when tuning a PID controller on line.
- Difficult for other types of controllers; need systematical methods

State of art

- Finite time stability:

$$x(t0)Rx(t0) < c1 \quad >>> \quad x(t)Rx(t) < c2$$

$$\text{for } t \text{ in } [t0, t0 + T],\ c1 < c2$$

- The system does not get control at all
- Latest FTS works/our works: x(t) can get smaller and smaller; not exact transient specifications
  - to do design
  - to consider output instead of state
  - to consider uncertain plants
  - to consider input constraints

- **Funnel Control, introduced by Ilchmann, et al. in 2002, [1]**
  - the trajectory follows some nice channel/envelop such as decay exponential
  - applicable to various nonlinear systems
  - but generally requires them to have low relative degree
  - not exact transient specifications
- **Prescribed performance functions, introduced by Bechlioulis, et al. 2008, [2].**
  - impose time-varying constraints as state performance requirement
  - define transformation to get unconstrained system
  - stabilise the new system
  - not exact transient specifications

- **Target Tubes : Bersekas & Rhodes, [3] and Glover & Sweppe, [4] from 70's.**
  - Considered similar ideas, with ellipsoidal sets on the state
  - allows backward computation of feedbacks
- **Tube-Based Robust MPC : Mayne,[5], Rakovic, [6], Cannon [7], from 2000's and later**
  - Find robust invariant set
  - At each time instant, find a sequence that specifies the centres and scalings of this set over a horizon.

Limitations

- Nominal system; not uncertain system
- Ellipsis case: norm based; not exact transient specifications
- State performance and feedback; not output
- Analytical solutions; stability alike
- No disturbance
- No input constraint

Objectives

- This work aims at developing a new control theory to ensure transient control performance. Given a system initiating within some set, we want to solve the problem of driving the state/output to a desired target set while satisfying time-varying performance constraints for all time over a given time horizon. Its theory and algorithms will be explored to find such solutions, see the details in the method section. The solutions will be tested on typical industrial examples in benchmark against the existing methods to show performance insurance and improvement. Our project is expected to deliver a new branch of control theory. The new theory can match actual requirements in control applications, Furthermore, The new theory can result in a new technology of control design and implementation for wide applications in industry.
- 10 years ago > Last year: math PhD from France
- My presentation here

2. The proposed approach

Overview

- Time domain, discrete-time and state space control theory
- A nominal system: x0, one point in 2D, moves to another point with a specific u; then to a set with a control set U
- Uncertain initial state: x0 is a set, *Polyhedral/多面体 set,* then a set moves to another
- Uncertain system: A and B are sets, polytypic/多面体 systems; convex hull of several fixed matrices
- Disturbance: D is a point, and a set
- Total state transition: the initial state set is mapped ("multiplied') by a system mapping group, shifted by a disturbance set, and controlled by the input set U
- *Polyhedral control p*erformance specifications on output
- *Polyhedral* constraint on input
- Numerical solution based on linear programming
- Demonstration by examples

Consider linear polytopic difference inclusions:

$$x(k+1) = \boldsymbol{A}(k)x(k) + \boldsymbol{B}(k)u(k)$$

$$y(k) = Cx(k)$$

$[\boldsymbol{A}(k)\ \boldsymbol{B}(k)] \in \mathrm{co}\{[A_i\ B_i]\}s$

$\mathrm{P}(M,m) = \{x : Mx \leq m\}$

X0 = P(Q, 0), initial set.

XT = P(Q, T), target set.

H(k) = P(Q, (k)), performance specs.

Cast performance on y to constraint on x:

$y(t) <= h_i(t);\ \ Cx(t) <= h_i(t);$

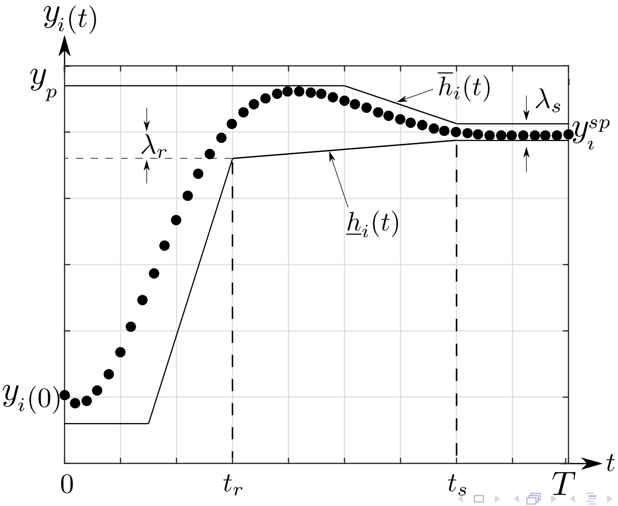
Problem Formulation

Consider :

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Dv(k) \\ y(k) &= Cx(k) \end{aligned}$$

- Let $\mathcal{P}(M, m) = \{x : Mx \leq m\}$
- $\mathcal{X}_0 = \mathcal{P}(Q, \psi_0)$, initial set.
- $\mathcal{X}_T = \mathcal{P}(Q, \psi_T)$, target set.
- $\mathcal{H}(k) = \mathcal{P}(Q, \phi(k))$, performance specs.
- Assume $v(k) \in \mathcal{V}(k) = \mathcal{P}(V, \gamma(k))$.
- $x^{(\hat{u}, \hat{v}, x_0)}(k)$: the solution at k, x_0 initial condition, $\hat{u}$ open-loop control function, $\hat{v}$ realisation of the disturbance both defined over $[0, k-1]$.

Traditional Step Response Specs.



Problem Statement

$$x(k+1) = Ax(k) + Bu(k) + Dv(k) \quad (1)$$
$$y(k) = Cx(k) \quad (2)$$

Problem Statement : Given the system (1)-(2) along with a time horizon, i.e., $k \in 0, ..., K, K \in \mathbb{Z}_{\geq 0}$; an initial set $\mathcal{X}_0$; a target set $\mathcal{X}_T$; and a time-varying set $\mathcal{H}(k)$ satisfying $\mathcal{X}_0 \subset \mathcal{H}(0)$, $\mathcal{X}_T = \mathcal{H}(K)$, find a time-varying state-feedback (output-feedback), $u(k, x(k))$ ($u(k, y(k))$) such that for all $x_0 \in \mathcal{X}_0$ and all v that satisfies $v(k) \in \mathcal{V}(k)$, we have that $x^{(\bar{u},v,x_0)}(k) \in \mathcal{H}(k)$ for all $k \in \{0, \ldots, K\}$.

Reachable Sets

Definition (one-step reachable set)

Consider :

$$x(k+1) = Ax(k) + Bu(k) + Dv(k) \tag{3}$$

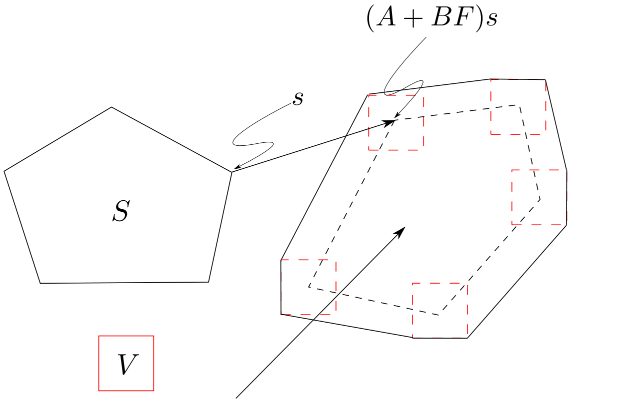
at time k along with a set $S \subset \mathbb{R}^n$. The *one-step reachable set* from S via equation (3) with $u(k) = \hat{u}(k)$ and $v(k) \in \mathcal{V}(k)$ is given by

$$\mathcal{R}^{(\hat{u}(k),\mathcal{V}(k))}(S) \triangleq \{x \in \mathbb{R}^n : \exists \hat{x} \in S, \exists \hat{v} \in \mathcal{V}(k) \\ \text{such that } x = A\hat{x} + B\hat{u}(k) + D\hat{v}\}.$$

- Let $u(k) = F(k)x(k)$, then :

$$\mathcal{R}^{(u(k),\mathcal{V}(k))}(S) = (A + BF(k))S \oplus D\mathcal{V}(k).$$

Example

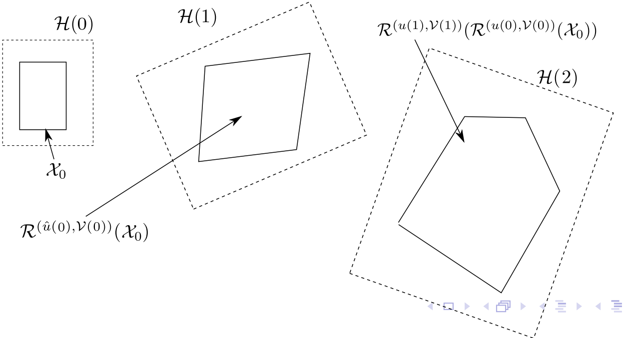


$$\mathcal{R}^{(u(k),V)}(S) = (A+BF)S \oplus DV$$

- Specifying $u(k) = F(k)x(k)$ at each k allows you to have more freedom in the shaping of $\mathcal{R}^{(u(k),\mathcal{V}(k))}(S)$

Overview

- Iterative estimation of reachable sets.
- Given $\mathcal{X}_0$, find $F(0)$ s.t. $u(0) = F(0)x(0)$ results in $\mathcal{R}^{(u(0),\mathcal{V}(0))}(\mathcal{X}_0) \subset \mathcal{H}(1)$.
- If successful, find $F(1)$ s.t. $u(1) = F(1)x(1)$ results in $\mathcal{R}^{(u(1),\mathcal{V}(1))}(\mathcal{R}^{(u(0),\mathcal{V}(0))}(\mathcal{X}_0)) \subset \mathcal{H}(2)$.
- etc...



Main Result

- A new way of over-estimating the one-step reachable set.

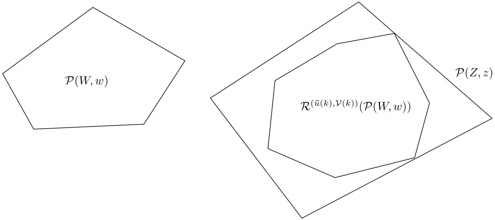
Theorem

Consider two polyhedral sets, $\mathcal{P}(W, w)$ *and* $\mathcal{P}(Z, z)$*, along with the system* (1) *at an arbitrary* $k \geq 0$ *with* $x(k) \in \mathcal{P}(W, w)$*,* $v(k) \in \mathcal{P}(V, \gamma(k))$ *and* $\bar{u}(k, x(k)) = F(k)x(k)$*. The following holds :* $\mathcal{R}^{(\bar{u}(k), \mathcal{V}(k))}(\mathcal{P}(W, w)) \subset \mathcal{P}(Z, z)$ *if and only if there exists a matrix* $G(k) \geq 0$*, that satisfies :*

$$G(k) \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix} = Z[A + BF(k) \quad D], \tag{4}$$

$$G(k) \begin{pmatrix} w \\ \gamma(k) \end{pmatrix} \leq z. \tag{5}$$

Proof Sketch (only if)



- Look at $x(k+1) = Ax(k)$
- $x(k) \in \mathcal{P}(W,w)$ and $x(k+1) \in \mathcal{P}(Z,z)$
- $\mu_i = \max_{x(k)} Z_i Ax(k)$ s.t. $Wx(k) \leq w$
- Dual : $\mu_i = \min_{g_i} g_i^T w$ s.t. $g_i^T W = Z_i A$, $g_i^T \geq 0$
- Let G_i be a solution to dual.
- $\mu_i = \max Z_i Ax \leq z_i$. But $\mu_i = G_i w$, thus $G_i w \leq z_i$
- Let G be matrix formed by stacking the solutions.
- Consider $x(k+1) = \left[\begin{array}{cc} A + BF(k) & D \end{array}\right] [x(k)\ v(k)]^T$

Algorithm

- Introduce : $\mathcal{X}(k) = \mathcal{P}(Q, \psi(k))$
- Recall : $\mathcal{H}(k) = \mathcal{P}(Q, \phi(k))$

Optimisation Problem

(OP1)

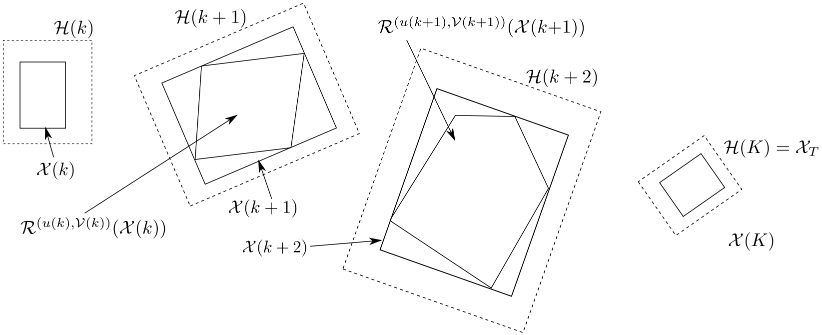
$$\begin{aligned} \min \quad & \mathcal{J}(\psi(k+1)) \\ \text{s.t.} \quad & G(k)\begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & V(k) \end{pmatrix} = Q[A + BF(k) \quad D], && (6) \\ & G(k) \geq 0, && (7) \\ & G(k)\begin{pmatrix} \psi(k) \\ \gamma(k) \end{pmatrix} \leq \psi(k+1), && (8) \\ & \psi(k+1) \leq \phi(k+1). && (9) \end{aligned}$$

- Inequality (9) is a new constraint. Ensures $\mathcal{X}(k) \subset \mathcal{H}(k)$.

Algorithm

Inputs *:* ψ_0, $V(k)$ *;* $\gamma(k)$, $\mathcal{J}(\psi(k+1))$ *for* $k = 0, \ldots, K-1$ *;*
$Q(k)$, $\phi(k)$ *for* $k = 0, \ldots, K$
Outputs *:* $\psi(k+1)$, $F(k)$ *for* $k = 0, \ldots K-1$*, if a solution exists.*

for $k = 0, \ldots, K-1$ ***do***
    attempt to find a solution to OP1
    if *no solution exists to OP1* ***then***
        end algorithm (without success)
    else if *a solution exists to OP 1* ***then***
        save $F(k)$ *and* $\psi(k+1)$
        if $k = K-1$ ***then***
            end algorithm (with success)
        end if
    end if
end for



$$\mathcal{R}^{(\bar{u}(k,x(k)),\mathcal{V}(k))}(\mathcal{X}(k)) \subset \mathcal{X}(k+1) \subset \mathcal{H}(k+1).$$

Remark

The optimisation problem can always be rendered feasible by the inclusion of q slack variables, $\rho_i(k+1) \geq 0$, $i = 1, \ldots, q$ *as follows : replace the constraints* (9) *with* $\psi_i(k+1) - \phi_i(k+1) \leq \rho_i(k+1)$ *for* $i = 1, \ldots, q$, *and replace the cost function with* $J(\psi(k+1)) + \alpha \mathbf{1}^T \rho(k+1)$, *where* $\rho(k+1) \triangleq (\rho_1(k+1), \rho_2(k+1), \ldots, \rho_q(k+1))^T$ *and* α *is some large positive number. This renders the performance constraints "soft".*

Discussion

Attractive features :

- If algorithm executes successfully, obtain $F(k)$, which works for *any* $x_0 \in \mathcal{X}_0$.
- Solving linear programs is computationally light, done off-line.

Negative features :

- No way of knowing if algorithm will execute successfully.
- Wrapping effect : over-estimates grow with time.
- However, this adds robustness, if algorithm executes successfully.
- We can't include control constraints (for now).

Mass-spring-damper

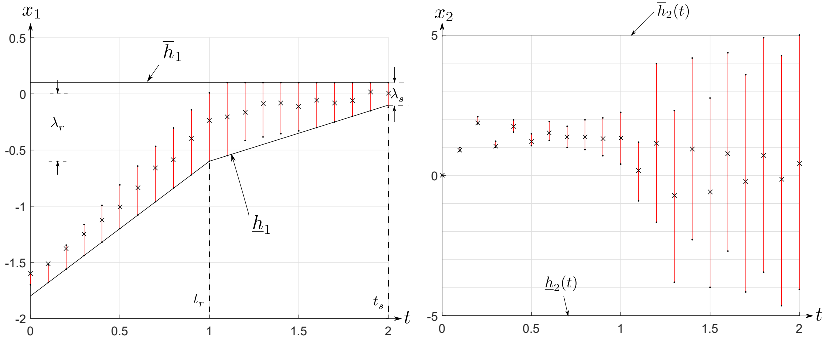
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v,$$

- $\mathcal{X}_0 = [-1.6, -1.7] \times [0.01, 0.01]$
- $\mathcal{X}_T = [-0.1, 0.1] \times [-5, 5]$
- $v(k) \in [-0.03, 0.03] \times [-0.03, 0.03]$ for $k < 13$, and $v(k) \in [-0.01, 0.01] \times [-0.01, 0.01]$ for $k \geq 13$

$$Q(k) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \ \psi_0 = \begin{bmatrix} -1.6 \\ 1.7 \\ 0.01 \\ 0.01 \end{bmatrix}, \ \psi_T = \begin{bmatrix} 0.1 \\ 0.1 \\ 5 \\ 5 \end{bmatrix}$$

$$\phi(k) = \begin{bmatrix} \overline{h}_1(kT_s) \\ \underline{h}_1(kT_s) \\ 5 \\ 5 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \gamma(k) = \begin{cases} (0.03)\mathbf{1}, & k < 13 \\ (0.01)\mathbf{1}, & k \geq 13 \end{cases}$$

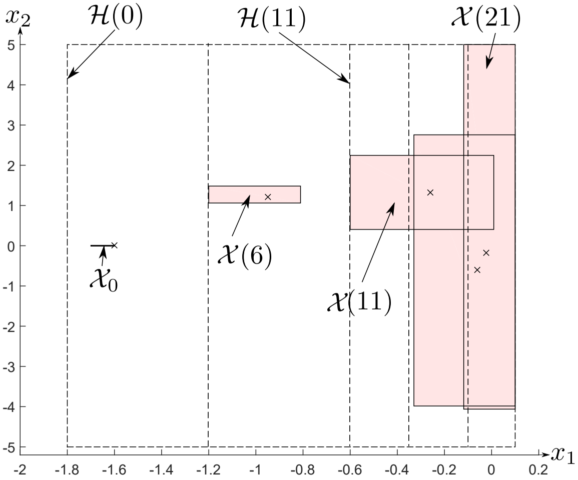
Mass-spring-damper



$$\mathcal{J}(\psi(k+1)) = \mathbf{1}^T\psi(k+1)$$

If no solution, $\mathcal{J}(\psi(k+1)) = \mathbf{1}^T\psi(k+1) + 100(\mathbf{1}^T\rho(k+1))$

Mass-spring-damper



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