Design of iterative learning control algorithms using a repetitive process setting and the generalized KYP lemma

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Introduction to KYP lemma

Facts from classical (1-D) control theory

- Frequency domain inequalities (FDI) have played **a crucial role** in describing design specification for feedback control synthesis (e.g. based on Bode or Nyquist plots).
- Due to the infinite dimensionality, however, FDIs are **not directly** useful for rigorous analysis and design of control systems.

Problem

How FDIs can be formulated in the mathematical form and if it is possible to convert them into numerically tractable procedures?

Introduction to KYP lemma

For linear, time invariant (LTI) systems, generally FDI can be formulated as

$$G(j\omega)^*\Pi G(j\omega) < 0, \ \ \forall\omega \in \mathbb{R}$$

where Π is a real symmetric matrix and

$$G(s) = C(sI - A)^{-1}B + D$$

is a matrix valued, real-rational transfer function.

Problem

The considered inequality contains the frequency variable $\Leftrightarrow$ infinite number of inequalities.

KYP lemma

Let $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times r}$, $\Theta \in \mathbb{H}^{n+r}$. If $\det(j\omega I - A) \neq 0$ for any $\omega \in \mathbb{R}$ then the following two statements are equivalent

- for any $\omega \in \mathbb{R} \cup \infty$

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \prec 0$$

- there exists a symmetric matrix P such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0.$$

Proof details

A. Rantzer: *On the Kalman-Yakubovich-Popov lemma*. Systems and Control Letters 28(1):7–10, 1996.

KYP lemma, cont'd

Main features

- **The infinitely many inequalities** parameterized by ω can be checked by solving **finite-dimensional** convex feasibility problem.
- Appropriate choices of Θ allows us to represent various system properties including positive-realness and bounded-realness.
- This standard KYP lemma treats FDIs for **the entire frequency range** only and it is not completely compatible with practical design specifications given in **the finite frequency range**.

Examples for FDIs specification

Open-loop shaping

Given a SISO plant $P(s)$, a set of specifications on the controller $K(s)$ is given in terms of the Nyquist plot of the open-loop transfer function

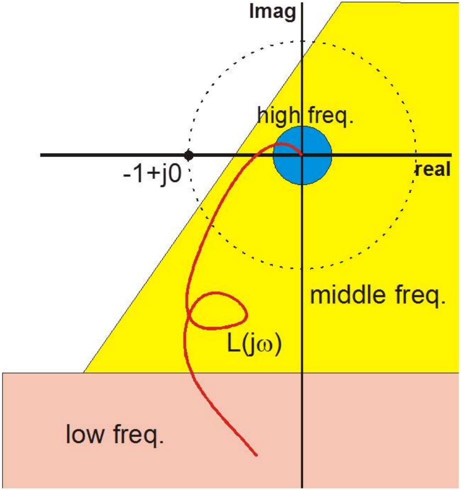
$$L(s) := K(s)P(s)$$

to meet design requirements:

- the high-gain in the **low frequency** range for the sensitivity reduction and reference tracking.
- to ensure stability margins and bandwidth maximization in the **middle-frequency** range,
- the small gain (i.e. roll-off) in the **high frequency** range for robust stability.

Examples for FDIs specification

Open-loop shaping, cont'd



Existing solutions

There are two main approaches to solve control problems in finite frequency:

1. A low/band/high-pass filter would be added to the system in series as a weight that emphasizes a particular frequency range and then the design parameters are chosen such that the weighted system norm is small. The deficiencies are:
   - the system complexity (e.g., controller order) is increased,
   - the process of selecting appropriate weights is tedious and can be time-consuming.
2. The frequency axis griding - FDIs are approximated by a finite number of FDIs at selected frequency points. However, it is difficult to:
   - determine a priori how fine the grid should be to achieve a certain performance.
   - impose performance guarantee in the design process as the violation of the specifications may occur at a frequency between grid points.

Towards generalized KYP lemma

Frequency set characterization

The frequency set is characterized by a quadratic equation and inequality of the form

$$\Lambda(\Phi,\Psi) := \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Phi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geqslant 0 \right\}$$

By an appropriate choice of Φ and Ψ, the set Λ can be specialized to define a certain range of the frequency variable λ. For example, for the continuous-time setting and low frequency range $\lambda \in \Lambda_{LF}$, i.e.

$$\Lambda_{LF} := \{ j\omega | \omega \in \mathbb{R}, |\omega| \leqslant \varpi \}$$

we have

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} -1 & 0 \\ 0 & \varpi^2 \end{bmatrix}$$

Generalized KYP lemma

Let Π, Φ, Ψ be given and let $G(\lambda)$ be rational function

$$G(\lambda) := C(\lambda I - A)^{-1}B + D$$

then the parameterized inequality condition

$$G(\lambda)^* \Pi G(\lambda) < 0, \ \ \forall \lambda \in \Lambda(\Phi, \Psi)$$

holds if and only if there exist matrices P, $Q \succ 0$ such that

$$\left[\begin{array}{cc} A & B \\ I & 0 \end{array}\right]^* (\Phi \otimes P + \Psi \otimes Q) \left[\begin{array}{cc} A & B \\ I & 0 \end{array}\right] + \Theta \prec 0$$

Proof details

T. Iwasaki, S. Hara: *Generalized KYP lemma: unified frequency domain inequalities with design applications*. IEEE Trans. on Automatic Control, 50(1), 41-59, 2005.

Finite frequency specification

It is routine to show that

$$|G(j\omega)| < \gamma, \quad \forall \omega_l \leqslant \omega \leqslant \omega_h,$$

is equivalent to

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^T \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix}}^{\Pi} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \prec 0, \quad \forall \omega_l \leqslant \omega \leqslant \omega_h,$$

and according to GKYP lemma we have

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \overbrace{\begin{bmatrix} -Q & P+j\omega_c Q \\ P+j\omega_c Q & -\omega_l\omega_h Q \end{bmatrix}}^{\Phi\otimes P+\Psi\otimes Q} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0,$$

where $\omega_c = (\omega_l + \omega_h)/2$.

Can we apply 1-D (G)KYP Lemma for solving some ILC problems?

Iterative learning control

Definition

Iterative learning control (ILC) is based on the notion that the performance of a system that executes the same task multiple times can be improved by learning from previous executions (or trials, iterations, passes).

film

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Iterative learning control

A common application area is industrial robotics such as a gantry robot executing a ‘pick and place’ task in synchronization with a moving conveyer, where the sequence of tasks is

- collect an object from a prescribed location,
- transfer it over a finite duration,
- place it on the conveyor,
- return to the original location and,
- repeat 4 first steps as many times as required.

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Practical industrial applications of ILC

- Industrial robots
  (S. Arimoto (1984), R.W. Longman (1994), M. Norrlow (2002))
- Rehabilitation robots
  (Z. Cai, D. Tong, E. Rogers (2010))
- Computer numerical control (CNC) machine tools
  (D.-I. Kim, S.Kim (1996))
- Wafer stage motion systems
  (D. de Roover, O.H. Bosgra (2000), B.G. Dijkstra (2003))
- Semibatch chemical reactors
  (M. Mezghani, G.Roux (2002))
- Advanced filtering and signal processing
  (H. Elci, R.W. Longman, M.Q. Phan (2002))

Learning versus non-learning control strategies

Non-learning control

- the same tracking error on each trial,
- previous iterations are information rich but they are unused.

Iterative learning control

- information from previous iterations are used,
- control performance is successively improved,
- high-performance tracking control can be achieved ,
- repetitive part of the error can be compensated.

Problem Setup

- Given system

$$x_k(p+1) = Ax_k(p) + Bu_k(p)$$
$$y_k(p) = Cx_k(p),\ 0 \leqslant p \leqslant \alpha < \infty$$

- **Control aim**: force $e_k(p) = r(p) - y_k(p)$ to converge (to zero) in the trial-to-trial direction

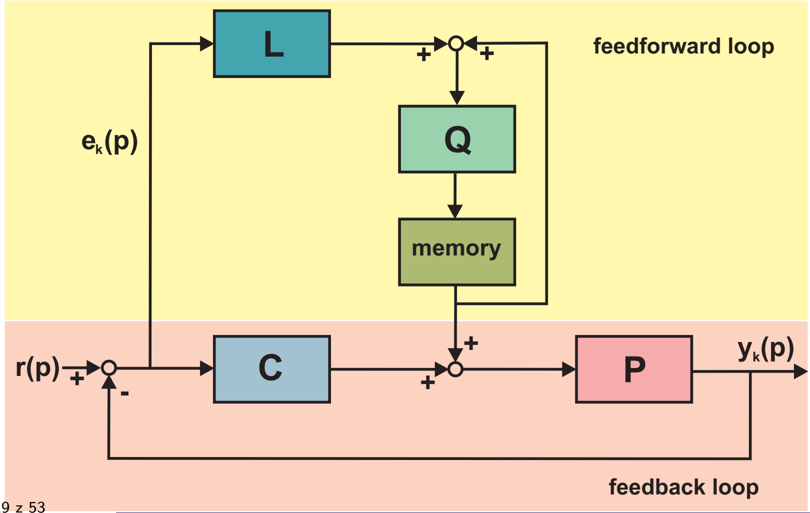
$$\lim_{k\to\infty} \|e_k\| = 0,$$

- Control law uses previous trial information, e.g.

$$u_{k+1}(p) = u_k(p) + \Delta u_k(p)$$

$\Delta u_k(p)$ – modification based on **the previous trial input**.

Schematic representation of ILC



Finite vs. infinite trial length

Some problems can be overcame by considering the trial length to be infinite, and then

- it enables the use of frequency domain analysis;
- if the trial length is long compared to time constant of the the system, infinite trial length is a good approximation of the real situation;
- a learning algorithm that converges of infinite trial length, must also converge on finite trial length;
- it gives insight in the properties of the controller which may be hard to obtain by finite time analysis.

Frequency domain analysis

Simple block diagram algebra gives

$$e_{k+1}(p) = Q(1 - LS_P)e_k(p),$$

where S_P denotes the sensitivity function

$$S_P = \frac{P}{1 + PC}.$$

Also it follows immediately that error convergence occurs if

$$\mathbf{|Q(1 - LS_P)| < 1, \forall |z| = 1}$$

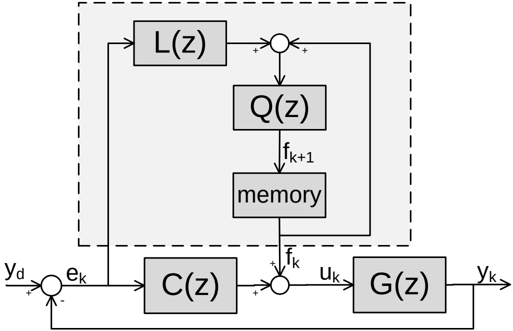
Problems:

- hard to find L such that $L = S_P^{-1}$,
- dynamics along the trial (transient response) is not considered.

ILC with current trial feedback

Practical problems

- learning controller is not able to compensate for random disturbances,
- a plant is unstable.



ILC with current trial feedback, cont'd

Based on the block diagram

$$F_{k+1}(z) = Q(z)\left(F_k(z) + L(z)E_k(z)\right)$$

and

$$E_k(z) = (I + G(z)C(z))^{-1}Y_d(z) - (I + G(z)C(z))^{-1}G(z)F_k(z)$$

Hence, the previous trial error feedforward contribution (assuming $Y_d(z) = 0$) to the current trial error is

$$E_k(z) = -\left[(I + G(z)C(z))^{-1}G(z)\right]F(z) = -S_P(z)F_k(z)$$

where $S_P(z) = (I + G(z)C(z))^{-1}G(z)$ denotes the sensitivity function and the propagation of the error from trial-to-trial is

$$E_{k+1}(z) = Q(z)\left(I - S_P(z)L(z)\right)E_k(z)$$

ILC with current trial feedback, cont'd

for ease of notation introduce

$$M(z) = Q(z)\left(I - S_P(z)L(z)\right)$$

and hence

$$E_{k+1}(z) - E_k(z) = M(z)\left(E_k(z) - E_{k-1}(z)\right)$$

Hence the trial-to-trial error converges monotonically in k provided

$$\|M(z)\|_\infty \triangleq \sup_{\omega\in[-\pi,\pi]} \overline{\sigma}(M(\mathrm{e}^{\mathrm{j}\omega})) < 1$$

and minimizing $\|M(z)\|_\infty$ increases the convergence speed.

ILC with current trial feedback, cont'd

General remarks

- Perfect tracking, i.e. $e_\infty(p) = 0, 0 < p \leqslant \alpha$ is only achieved when $Q(z) = I$.
- This choice is, however, prone to the effects of high frequency noise and non-repeating disturbances. Hence, Q has to be chosen as a low-pass filter.
- Perfect tracking of the reference is achieved in the specified frequency range and attenuated over the remainder.
- Q-filter cut-off frequency should be equal or larger than the desired close-loop bandwidth.
- Fast monotonic convergence will result if $L(z) = S_p^{-1}(z)$.
- Also the feedback controller C must ensure that $(I + G(z)C(z))^{-1} \approx 0$.

General control problem formulation

Assume $Q = I$ and then

$$M(z) = I - G(z)\begin{bmatrix} C(z) & L(z) \end{bmatrix}\left(\begin{bmatrix} (I + G(z)C(z)) & G(z)L(z) \\ 0 & I \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}.$$

$M(z)$ can be represented as a general control configuration, where the generalized plant P, i.e. the interconnection system of the controlled system, is

$$P(z) = \left[\begin{array}{c|c} P_{11}(z) & P_{12}(z) \\ \hline P_{21}(z) & P_{22}(z) \end{array}\right] = \left[\begin{array}{c|c} I & -G(z) \\ \hline 0 & -G(z) \\ I & 0 \end{array}\right],$$

or

$$M(z) = P_{11}(z) + P_{12}(z)K(z)\left(I - P_{22}(z)K(z)\right)^{-1}P_{21}(z),$$

where $K(z) = [C(z) \;\; L(z)]$.

The general plant (P) is represented by

$$\begin{aligned} x_{k+1}(p+1) =& Ax_{k+1}(p) + Bu_{k+1}(p), \\ \overline{y}_{k+1}(p) =& -Cx_{k+1}(p), \\ \widehat{y}_{k+1}(p) =& e_k(p), \\ e_{k+1}(p) =& -Cx_{k+1}(p) + e_k(p). \end{aligned}$$

Also, suppose K is represented by

$$\begin{aligned} \widetilde{x}_{k+1}(p+1) =& A_K\widetilde{x}_{k+1}(p) + \begin{bmatrix} B_{K1} & B_{K2} \end{bmatrix} \begin{bmatrix} \overline{y}_{k+1}(p) \\ \widehat{y}_{k+1}(p) \end{bmatrix}, \\ u_{k+1}(p) =& C_K\widetilde{x}_{k+1}(p) + \begin{bmatrix} D_{K1} & D_{K2} \end{bmatrix} \begin{bmatrix} \overline{y}_{k+1}(p) \\ \widehat{y}_{k+1}(p) \end{bmatrix}, \end{aligned}$$

which results in the controller realization
$K(z) = D_K + C_K(zI - A_K)^{-1}B_K$ where

$$B_K = \begin{bmatrix} B_{K1} & B_{K2} \end{bmatrix},\ D_K = \begin{bmatrix} D_{K1} & D_{K2} \end{bmatrix}.$$

ILC as linear repetitive process

The ILC dynamics can now be written as

$$\left[\begin{array}{c} x_{k+1}(p+1) \\ \widetilde{x}_{k+1}(p+1) \\ \hline e_{k+1}(p) \end{array}\right] = \left[\begin{array}{cc|c} A-BD_{K1}C & BC_K & BD_{K2} \\ -B_{K1}C & A_K & B_{K2} \\ \hline -C & 0 & I \end{array}\right] \left[\begin{array}{c} x_{k+1}(p) \\ \widetilde{x}_{k+1}(p) \\ \hline e_k(p) \end{array}\right].$$

which is a discrete linear repetitive process state-space model with current trial state vector $[x_{k+1}^{\top}\ \widetilde{x}_{k+1}^{\top}]$, zero input vector and previous pass profile e_k.

The main problem: How to deal with very slow convergence since $e_{k+1} = -Cx_{k+1}(p+1) + e_k(p)$. Also we have a problem when the relative degree of a plant is > 1.

Since our plant has relative degree r, the previous trial error is shifted by r samples to form an the anticipative feedforward control law.

- The signal $f_k(p)$ at time instant p is paired with the error signal $e_k(p+r)$ at time instant $p+r$.
- Obviously, this is possible since the error signal from trial k is available once this trial is complete. As a consequence this anticipative control law, the learning controller is taken as $z^r L$ instead of L.

The generalized plant P resulting from the above modification is

$$P(z) = \left[\begin{array}{c|c} P_{11}(z) & P_{12}(z) \\ \hline P_{21}(z) & P_{22}(z) \end{array}\right] = \left[\begin{array}{c|c} I & -G(z) \\ \hline 0 & -z^r G(z) \\ I & 0 \end{array}\right],$$

where P_{22} includes anticipatory operator and the forward time shift is applied to the error signal transmitted through $\widehat{y}_{k+1}$.

ILC as linear repetitive process, cont'd

Since $(zI-\overline{A})(zI-\overline{A})^{-1}=I$, then

$$z(zI-\overline{A})^{-1}=I+\overline{A}(zI-\overline{A})^{-1}.$$

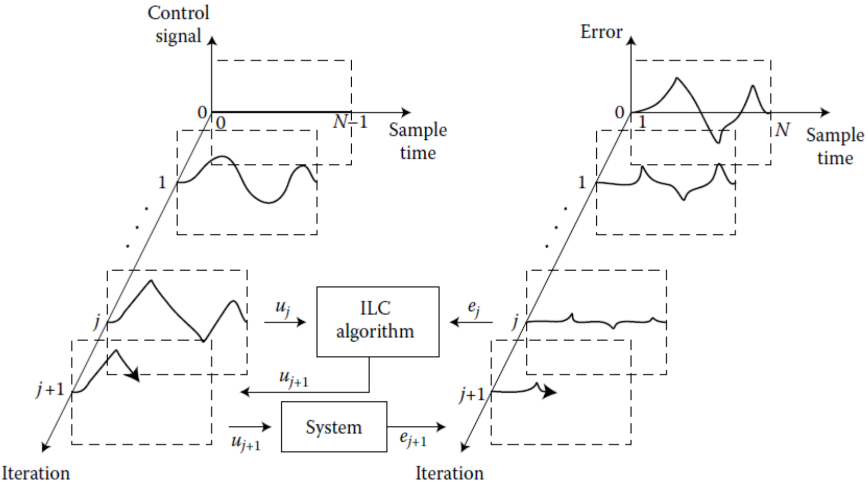
$M(z)$ can be rewritten as

$$M(z)=\mathbb{C}(zI-\mathbb{A})^{-1}\mathbb{B}_0+\mathbb{D}_0,$$

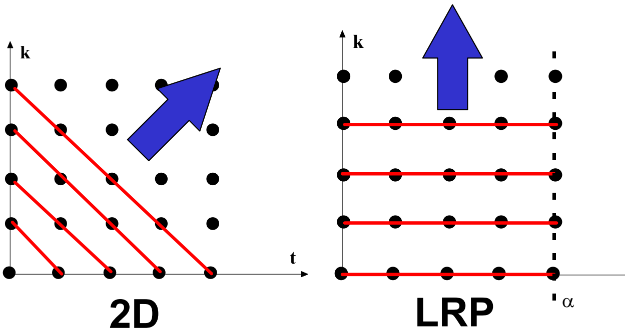
where the matrices in the state-space quadruple $\{\mathbb{A},\mathbb{B}_0,\mathbb{C},\mathbb{D}_0\}$ are

$$\begin{aligned}
\mathbb{A} &= (\mathcal{A}+\mathcal{B}\mathcal{K}) = \begin{bmatrix} A-BD_{K1}C & BC_K \\ -B_{K1}C & A_K \end{bmatrix},\ \mathbb{B}_0 = \begin{bmatrix} BD_{K2} \\ B_{K2} \end{bmatrix},\\
\mathbb{D}_0 &= \overline{D}_0+\overline{C}\mathcal{A}^{r-1}\overline{B}_0 = I-CA^{r-1}BD_{K2},\\
\mathbb{C} &= \overline{C}\mathcal{A}^{r-1}(\mathcal{A}+\mathcal{B}\mathcal{K}) = \begin{bmatrix} -CA^r+CA^{r-1}BD_{K1}C & -CA^{r-1}BC_K \end{bmatrix}.
\end{aligned}$$

ILC scheme as a repetitive process



2-D Systems and Linear Repetitive Processes



- their dynamics evolve in two separate directions,
- information along a given trial (pass), is only propagated over finite duration - the trial length.

Applying 2-D system theory

Define the shift operators $z_1,\ z_2$ in the along the trial (p) and trial-to-trial (k) directions respectively as

$$\begin{aligned}\eta_k(p) &:= z_1 \eta_k(p+1),\\ e_k(p) &:= z_2 e_{k+1}(p)\end{aligned}$$

Then the 2-D characteristic polynomial is

$$\rho(z_1, z_2) = \det\left(\begin{bmatrix} I - z_1\mathbb{A} & -z_1\mathbb{B}_0 \\ -z_2\mathbb{C} & I - z_2\mathbb{D}_0 \end{bmatrix}\right)$$

General problem

It is hard to check stability conditions which involve two variable polynomials (e.g. the number of poles can be infinite).

Stability theory of repetitive processes

The stability theory of repetitive processes consists of two distinct stability concepts

- **asymptotic stability**, that guarantees the existence of a limit profile which is described by a 1-D linear system state space model,
- **stability along the pass**, that guarantees the existence of a limit profile and ensures that the resulting limit profile is stable along the pass dynamics.

Facts

- In most cases, asymptotic stability is investigated through the use of 1-D system theory applied to the equivalent 1-D model,
- however, it turns out that asymptotic stability cannot guarantee that the resulting pass profile has 'acceptable' characteristic.

Stability along the pass

Theorem

Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then a discrete LRP is stable along the pass if and only if

i) $\rho(\mathbb{D}_0) < 1$,

ii) $\rho(\mathbb{A}) < 1$,

iii) *all eigenvalues of* $M(\mathrm{e}^{\mathrm{j}\omega}) = \mathbb{C}(\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A})^{-1}\mathbb{B}_0 + \mathbb{D}_0$, $\forall \omega \in [-\pi, \pi]$ *have modulus strictly less than unity.*

Facts

☺ It is necessary and sufficient condition

☺ Nice in theory

☹ Hard to apply

Stability along the pass

More facts...

- $\rho(\mathbb{D}_0) < 1$ - asymptotic stability condition
- $\rho(\mathbb{A}) < 1$ - the first pass profile is uniformly bounded with respect to the pass length.
- iii) means that each frequency component of the initial pass profile to be attenuated from pass to pass.
- In practise the initial pass profiles have finite frequency spectrums rather than in entire frequency domain.

Nyquist-based interpretation

For stable along the pass LRPs the Nyquist plot generated by (iii) lies inside the unit circle in the complex plane.

Stability for monotonic convergence

Tracking error converges as $k \to \infty$, iff

$$\rho\left(M(\mathrm{e}^{\mathrm{j}\omega})\right) < 1,\ \forall \omega \in [-\pi, \pi].$$

Practical experience shows that some ILC laws have poor transients during the convergence process even if the above condition is satisfied (e.g. the tracking error may grow over some number of trials). To avoid these problems, a stronger (monotonic) convergence criteria (but sufficient one) is used

$$\overline{\sigma}(M(\mathrm{e}^{\mathrm{j}\omega})) < 1,\ \forall \omega \in [-\pi, \pi].$$

The above condition is more practical and commonly used in practice. Due to the fact that

$$\|M(z)\|_\infty \triangleq \sup_{\omega \in [-\pi, \pi]} \overline{\sigma}(M(\mathrm{e}^{\mathrm{j}\omega})),$$

the convergence problem for ILC algorithms is reformulated as the $\mathcal{H}_\infty$ control problem.

ILC in repetitive process framework

LMI-based design procedure

Consider an ILC algorithm representation as an LRP. Then such a process is stable along the pass, i.e. the resulting ILC algorithm is monotonically convergent, if there exist $S \succ 0$, $P \succ 0$ and W such that

$$\mathbb{A}^T S \mathbb{A} - S \prec 0$$

$$\begin{bmatrix} -P & -W & 0 & 0 \\ -W^T & P + \mathbb{A}^T W + W^T \mathbb{A} & W^T \mathbb{B}_0 & \mathbb{C}^T \\ 0 & \mathbb{B}_0^T W & -I & \mathbb{D}_0^T \\ 0 & \mathbb{C} & \mathbb{D}_0 & -I \end{bmatrix} \prec 0$$

Some facts ...

- stability along the pass = monotonic convergence ;
- In practise the initial pass profiles have finite frequency spectrums rather than in entire frequency domain.

Application of KYP lemma

Taking $G(\mathrm{e}^{\mathrm{j}\omega}) = \mathbb{C}(\mathrm{e}^{\mathrm{j}\omega}I - \mathbb{A})^{-1}\mathbb{B}_0 + \mathbb{D}_0$, KYP lemma gives

$$\begin{bmatrix} G(\mathrm{e}^{\mathrm{j}\omega}) \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} G(\mathrm{e}^{\mathrm{j}\omega}) \\ I \end{bmatrix} \prec 0, \quad \forall \omega \in [-\pi, \pi],$$

and hence

$$G(\mathrm{e}^{\mathrm{j}\omega})^* G(\mathrm{e}^{\mathrm{j}\omega}) \prec I, \quad \forall \omega \in [-\pi, \pi],$$

Therefore we focus on

$$\overline{\sigma}(G(\mathrm{e}^{\mathrm{j}\omega})) < 1, \ \forall \omega \in [-\pi, \pi]$$

instead of true stability condition given by

$$\rho\left(G(\mathrm{e}^{\mathrm{j}\omega})\right) < 1, \ \forall \omega \in [-\pi, \pi]$$

Generalized KYP lemma

For $Q \succ 0$ and a symmetric matrix P

$$\begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix}^\top \Xi \begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathbb{C} & \mathbb{D}_0 \\ 0 & I \end{bmatrix}^\top \Pi \begin{bmatrix} \mathbb{C} & \mathbb{D}_0 \\ 0 & I \end{bmatrix} \prec 0,$$

where Ξ is specified as follows:

- for the low frequency range

$$\Xi = \left[\begin{array}{c|c} \Xi_{11} & \Xi_{12} \\ \hline \Xi_{12}^* & \Xi_{22} \end{array}\right] = \left[\begin{array}{c|c} -P & Q \\ \hline Q & P - 2\cos(\theta_l) Q \end{array}\right],$$

- for the middle frequency range

$$\Xi = \left[\begin{array}{c|c} \Xi_{11} & \Xi_{12} \\ \hline \Xi_{12}^* & \Xi_{22} \end{array}\right] = \left[\begin{array}{c|c} -P & \mathrm{e}^{\mathrm{j}(\theta_1+\theta_2)/2} Q \\ \hline \mathrm{e}^{-\mathrm{j}(\theta_1+\theta_2)/2} Q & P - (2\cos((\theta_2 - \theta_1)/2)) Q \end{array}\right],$$

- and for the high frequency range

$$\Xi = \left[\begin{array}{c|c} \Xi_{11} & \Xi_{12} \\ \hline \Xi_{12}^* & \Xi_{22} \end{array}\right] = \left[\begin{array}{c|c} -P & -Q \\ \hline -Q & P + 2\cos(\theta_h) Q \end{array}\right].$$

Solving synthesis problem via LMI

Assume that the matrix variable $\mathcal{W}$ and its inverse are partitioned into blocks as

$$\mathcal{W} = \begin{bmatrix} X & ? \\ U & ? \end{bmatrix}, \ \mathcal{W}^{-1} = \begin{bmatrix} N & ? \\ R & ? \end{bmatrix},$$

and define the transformation matrices F_1 and F_2 as

$$F_1 = \begin{bmatrix} X & I \\ U & 0 \end{bmatrix}, \ F_2 = \begin{bmatrix} I & N \\ 0 & R \end{bmatrix},$$

where $\mathcal{W}F_2 = F_1$ and introduce the notation

$$\widehat{W} = F_2^\top \mathcal{W} F_2 = \begin{bmatrix} X & I \\ Z^\top & N^\top \end{bmatrix}, \ \widehat{A} = F_2^\top \mathcal{W}^\top \mathbb{A} F_2 = \begin{bmatrix} X^\top A - \widetilde{B}_1 C & \widetilde{A} \\ A - BD_{K1}C & AN + B\widetilde{C} \end{bmatrix},$$

$$\widehat{B}_0 = F_2^\top \mathcal{W}^\top \mathbb{B}_0 = \begin{bmatrix} \widetilde{B}_2 \\ BD_{K2} \end{bmatrix}, \ \widehat{C} = \mathbb{C}F_2 = \begin{bmatrix} -CA^{r-1}(A - BD_{K1}C) & -CA^{r-1}(AN + B\widetilde{C}) \end{bmatrix},$$

where

$$\widetilde{A} = X^\top AN - X^\top BD_{K1}CN - U^\top B_{K1}CN + X^\top BC_K R + U^\top A_K R, \ Z = X^\top N + U^\top R,$$

$$\widetilde{B}_1 = X^\top BD_{K1} + U^\top B_{K1}, \ \widetilde{B}_2 = X^\top BD_{K2} + U^\top B_{K2}, \ \widetilde{C} = C_K R - D_{K1}CN.$$

ILC design in entire frequency range

Theorem

An ILC algorithm described as a discrete LRP is stable along the pass and hence monotonic trial-to-trial error convergence occurs if there exist matrices $\widehat{W}$, $\widetilde{A}$, $\widetilde{B}_1$, $\widetilde{B}_2$, $\widetilde{C}$, D_{K1}, D_{K2}, N, X, Z, $\widehat{S} \succ 0$ *and a symmetric matrix* $\widehat{\mathcal{P}}$ *such that the following LMIs are feasible*

$$\begin{bmatrix} \widehat{S} - \widehat{W} - \widehat{W}^\top & \widehat{A} \\ \widehat{A}^\top & -\widehat{S} \end{bmatrix} \prec 0$$

$$\begin{bmatrix} -\widehat{\mathcal{P}} & -\widehat{W} & 0 & 0 \\ -\widehat{W}^\top & \widehat{\mathcal{P}} + \widehat{A}^\top + \widehat{A} & \widehat{B}_0 & \widehat{C}^\top \\ 0 & \widehat{B}_0^\top & -I & \mathbb{D}_0^\top \\ 0 & \widehat{C} & \mathbb{D}_0 & -I \end{bmatrix} \prec 0$$

ILC design in entire frequency range, cont'd

Suppose that the LMIs are feasible. Then the following is a systematic procedure for obtaining the corresponding controller matrices:

1. Compute the singular value decomposition (SVD) of $Z - X^\top N$ to obtain square and invertible matrices U_1, V_1 such that $Z - X^\top N = U_1 \Sigma_1 V_1^\top$.
2. Choose the matrices U and R as $U^\top = U_1 \Sigma_1^{\frac{1}{2}}$, $R = \Sigma_1^{\frac{1}{2}} V_1^\top$.
3. Compute the matrices of the ILC controller state space model using

$$
\begin{aligned}
C_K =& (\widetilde{C} + D_{K1} C N) R^{-1}, \\
B_{K1} =& U^{-\top} (\widetilde{B}_1 - X^\top B D_{K1}), \\
B_{K2} =& U^{-\top} (\widetilde{B}_2 - X^\top B D_{K2}), \\
A_K =& U^{-\top} (\widetilde{A} - X^\top A N + X^\top B D_{K1} C N + U^\top B_{K1} C N - X^\top B C_K R) R^{-1},
\end{aligned}
$$

where $U^{-\top} = \left(U^{-1}\right)^\top = \left(U^\top\right)^{-1}$.

Finite frequency ILC scheme design

- The ILC algorithm can address various frequency specifications which are defined in chosen frequency ranges in order to ensure
  - good convergence rate (in low frequency)
  - low sensitivity to the sensor noise (in high frequency range)
- The choice of frequency ranges and their numbers have to be determined according to prescribed performance, convergence rate, and robustness.
- To relax our problem it is possible to divide frequency range into H intervals (not necessary with the same length) such that

$$[0, \pi] = \bigcup_{h=1}^{H} [\omega_{h-1}, \omega_h],$$

  where $\omega_0 = 0$ and $\omega_H = \pi$.
- The higher speed of monotonic convergence can be obtained via optimization procedure.

Finite frequency ILC scheme design, cont'd

Consider an ILC algorithm described as a discrete LRP. Furthermore, suppose that the entire frequency range is arbitrarily divided into H possible different frequency intervals. Then i) the resulting repetitive process is stable along the pass, ii) monotonic trial-to-trial error convergence occurs and iii) the finite frequency performance specifications are satisfied if there exist matrices $\widehat{W}$, $\widetilde{A}$, $\widetilde{B}_1$, $\widetilde{B}_2$, $\widetilde{C}$, D_{K1}, D_{K2}, N, X, Z, $\widehat{\mathcal{Q}}_h \succ 0$, $\widehat{S} \succ 0$, symmetric $\widehat{\mathcal{P}}_h$ and arbitrary chosen scalars μ_h such that the following LMIs are feasible

$$\begin{bmatrix} \widehat{S} - \widehat{W} - \widehat{W}^\top & \widehat{A} \\ \widehat{A}^\top & -\widehat{S} \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} -\widehat{\mathcal{P}}_h - \widehat{W} - \widehat{W}^\top & \mathrm{e}^{\mathrm{j}\omega_{ch}} \widehat{\mathcal{Q}}_h - \widehat{W} & 0 & 0 \\ \mathrm{e}^{-\mathrm{j}\omega_{ch}} \widehat{\mathcal{Q}}_h - \widehat{W}^\top & \widehat{\mathcal{P}}_h - 2\cos(\omega_{dh})\widehat{\mathcal{Q}}_h + \widehat{A}^\top + \widehat{A} & \widehat{B}_0 & \widehat{C}^\top \\ 0 & \widehat{B}_0^\top & -\mu_h^2 I & \mathbb{D}_0^\top \\ 0 & \widehat{C} & \mathbb{D}_0 & -I \end{bmatrix} \prec 0,$$

for all $h = 1, \ldots, H$, where

$$\omega_{ch} = \frac{\omega_{h-1} + \omega_h}{2}, \; \omega_{dh} = \frac{\omega_h - \omega_{h-1}}{2}.$$

Q-filter design procedure

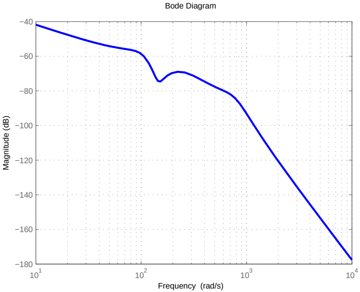
Some facts ...

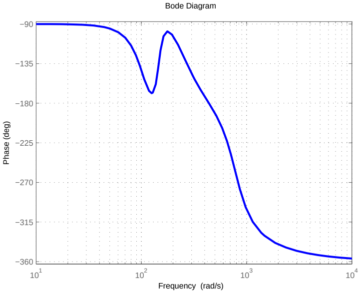
- It is evident that for low frequencies it is (relatively) easy to obtain $\overline{\sigma}(M(\mathrm{e}^{\mathrm{j}\omega})) < 1$ since in this frequency range the sensitivity function satisfies $|S_P(z)| \approx 0$.
- When feedback becomes less effective and $|S_P| \approx 1$, the learning filter L has to be chosen as the inverse of S_P.
- In particular, this means that L has to compensate unmodelled high frequency dynamics and obviously when S_P is strictly proper its exact inverse is improper and hence this not possible for all frequencies.
- Furthermore, it implies that the convergence can be achieved for $|Q(z)| \approx 1$ over the frequency range where L approximates S_P^{-1} well only. Therefore, the Q filter can be chosen as the low pass filter with cut-off frequency equal to ω_H, i.e. the highest frequency for which the result of the last Theorem is valid.

Multi-axis gantry robot

Gantry robot model

- Since the axes are orthogonal, it is assumed that there is minimal interaction between them.
- Each axis of is modeled based on frequency response function (FRF) method.
- X-axis dynamics is only considered here.





Gantry robot model

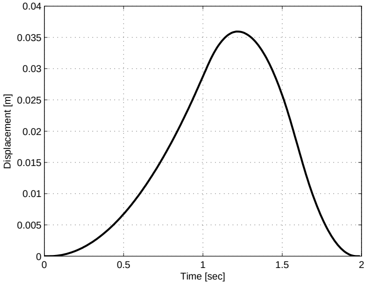
- 7th order continuous time transfer-function has been identified.
- Discretization with 100[Hz] results in the state space model for the X-axis

$$A = \begin{bmatrix} 0.3879 & 1.0000 & 0.2138 & 0 & 0.1041 & 0 & 0.0832 \\ -0.3898 & 0.3879 & 0.1744 & 0 & 0.0849 & 0 & 0.0678 \\ 0 & 0 & -0.1575 & 0.2500 & -0.2006 & 0 & -0.1603 \\ 0 & 0 & -0.3103 & -0.1575 & -0.0555 & 0 & -0.0444 \\ 0 & 0 & 0 & 0 & 0.0353 & 0.5000 & 0.2809 \\ 0 & 0 & 0 & 0 & -0.0164 & 0.0353 & -0.2757 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0910 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0391 & 0 & 0.0146 & 0 & 0.0071 & 0 & 0.0057 \end{bmatrix}.$$

Reference trajectory to be learnt

◆ Pick and place process



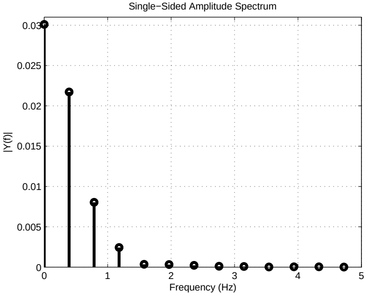


Figure: The reference trajectory for the X-axis. Corresponding frequency spectrum.

Controller design

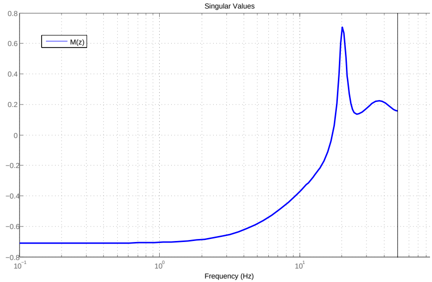
◆ Applying the design procedure with for two frequency ranges (0,2)Hz and (2,10)Hz gives the learning (L) and feedback controllers (C) as

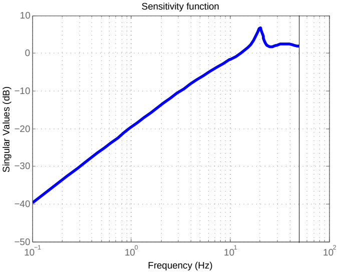
$$L(z)=\frac{78.92z^7-41.7z^6+24.97z^5-6.26z^4+4.65z^3-1.752z^2+0.155z-0.0124}{z^7+0.25z^6+0.4883z^5+0.3586z^4+0.0621z^3-0.09096z^2+0.00908z-0.000768},$$

$$C(z)=\frac{887.8z^6-694.4z^5+457.5z^4+45.55z^3+56.35z^2-3.314z+0.549}{z^7+0.25z^6+0.4883z^5+0.3586z^4+0.0621z^3-0.09096z^2+0.00908z-0.000768}$$

Simulation results

A simulation of the controlled system was performed and results are





The cut-off frequency of Q-filter has to be 10[Hz]

Concluding remarks

1. GKYP lemma, enables direct loop shaping through LMI optimization without frequency gridding or weights.
2. This allows us to solve many complex analysis and synthesis problems in 1-D/2-D system theory through LMI-based optimization.
3. A practical alternative is to use a 2-D systems setting where ILC can be represented in this form with one direction of information propagation from trial-to-trial and the other along the trial.
4. The presented approach for ILC scheme design offers advances not possible to obtain by other known results.