A perturbation based proportional integral extremum-seeking control technique

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Introduction

- Extremum-seeking control (ESC) has been the subject of considerable research effort over the last decade.
- Mechanism dates back to the 1920s [Leblanc, 1922]
  - Objective is to drive a system to the optimum of a measured variable of interest [Tan et al., 2010]
- Revived interest in the field was primarily sparked by Krstic and co-workers [Krstic and Wang, 2000]
  - Provided an elegant proof of the convergence of a standard perturbation based ESC for a general class of nonlinear systems

Introduction

Basic ESC objectives:

- Given an (unknown) nonlinear dynamical system and (unknown) measured cost function:

$$\begin{aligned} \dot{x} &= f(x,u) \qquad (1) \\ y &= h(x) \qquad (2) \end{aligned}$$

- The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$.

Basic ESC Loop

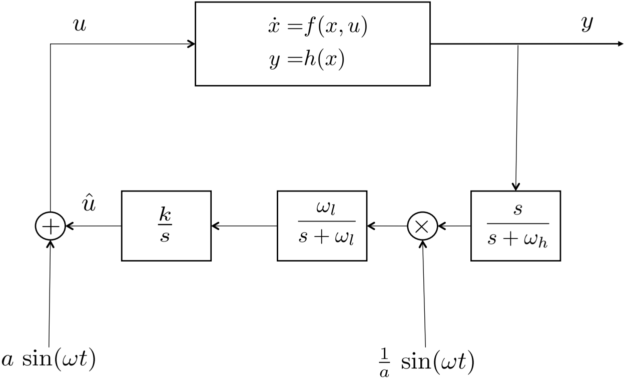


Figure: Standard ESC Loop.

Introduction

- The stability analysis [Krstic and Wang, 2000] relies on two components:
  1. an averaging analysis of the persistently perturbed ESC loop
  2. a time-scale separation of ESC closed-loop dynamics between the system dynamics and the quasi steady-state extremum-seeking task.
- This analysis shows that the tuning parameters of the ESC must be chosen very carefully to guarantee convergence to a neighbourhood of the unknown optimum.

Proportional Integral ESC

Limitations associated with the two time-scale approach to ESC remains problematic.

- Two (or more) time-scale assumption is required to ensure that optimization operates at a quasi steady-state time-scale
- Convergence is very slow.
- Limits applicability in practice.

Improvement in transient performance has been widely studied:

- An observer-based fast extremum seeking control approach is proposed in Moase and Manzie [2012]
- Newton-seeking Ghaffari et al. [2012], Moase et al. [2010], Liu and Krstic [2014]
- Lie-bracket averaging analysis Dürr et al. [2013]

Objectives

- The main objective of this study is to propose an ESC that removes the need for time-scale separation.
  - Technique utilizes a standard perturbation based approach.
- The proposed controller has two modes:
  - Proportional control
  - Integral control
- The main contribution is two-fold:
  - Minimize impact of time-scale separation on transient performance.
  - Achieve stabilization of unknown nonlinear system to optimum.

Problem Definition

- The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$.
  - The equilibrium (or steady-state) map is the n dimensional vector $\pi(u)$ which is such that:

$$f(\pi(u), u) = 0.$$

  - The equilibrium cost function is given by:

$$y = h(\pi(u)) = \ell(u) \tag{3}$$

  - The problem is to find the minimizer u^* of $y = \ell(u^*)$.

Proportional Integral ESC

Assumption 1

The cost $h(x)$ is such that

1. $\frac{\partial h(x^*)}{\partial x} = 0$

Assumption 2

The cost $h(x)$ has strong relative degree one.

By Assumption 2, the unknown dynamics can be decomposed as:

$$\begin{aligned} \dot{\xi} &= \phi(\xi, y) \\ \dot{y} &= L_f h + L_g h u \end{aligned}$$

where $\xi \in \mathbb{R}^{n-1}$, ϕ is a smooth vector valued function of ξ and $y = h(x)$.

Proportional Integral ESC

Assumption 3

The normal form dynamics are such that:

- there exists a function $W(\xi)$ such that:

$$\beta_1 \|x - \pi(\hat{u})\|^2 \leq W(\xi) + h - h(\pi(\hat{u})) \leq \beta_2 \|x - \pi(\hat{u})\|$$

  for positive constants β_1, β_2 and there exists a nonnegative constant k^* such that

$$\frac{\partial W}{\partial \xi}\phi(\xi, y) + L_f h - k^* \left\|L_g h\right\|^2 + L_g h \hat{u} \leq -\alpha_3 \|x - \pi(\hat{u})\|^2$$

  for a positive constant $\alpha_3 > 0$, $\forall x \in \mathcal{D}(u^*)$ and $\forall \hat{u} \in \mathcal{U}$.

- This describes a class of *minimum phase stabilizable* nonlinear systems.

Proportional Integral ESC

Assumption 4

The equilibrium steady-state map $\ell(u)$ is such that

$$\nabla_u \ell(u)(u - u^*) \geq \alpha_u \|u - u^*\|^2$$

for some positive constant α_u $\forall u \in \mathcal{U}$.

- Local convexity of the steady-state cost around u^*.

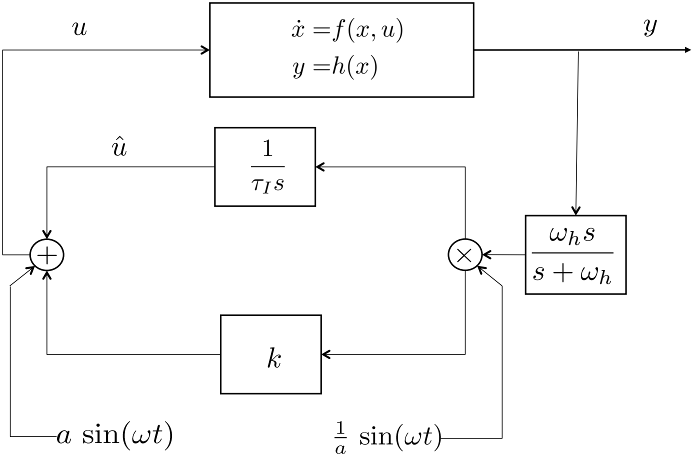
Proportional Integral ESC

- Proposed PI-ESC algorithm:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
\dot{v} &= -\omega_h v + y \\
\dot{\hat{u}} &= -\frac{1}{\tau_I}(-\omega_h^2 v + \omega_h y)\sin(\omega t) \\
u &= -\frac{k}{a}(-\omega_h^2 v + \omega_h y)\sin(\omega t) + \hat{u} + a\sin(\omega t).
\end{aligned}$$

- Tuning parameters:
  - k and τ_I are the proportional and integral gain
  - a ω are the dither amplitude an frequency
  - $\omega_h(>> \omega)$ is the high-pass filter parameter.

Proportional Integral ESC



Proportional Integral ESC

Theorem 1

Consider the nonlinear closed-loop PIESC system with cost function $y = h(x)$. Let Assumptions 1, 2, 3 and 4 hold. Then

1. there exists a τ_I^* such that for all $\tau_I > \tau_I^*$ the trajectories of the nonlinear system converge to an $\mathcal{O}(1/\omega)$ neighbourhood of the unknown optimum equilibrium, $x^* = \pi(u^*)$,
2. there exists $\omega^* > 0$ such that, for any $\omega > \omega^*$, the unknown optimum is a practically stable equilibrium of the PIESC system with a region of attraction whose size grows with the ratio $\frac{a}{k}$,
3. $\|x - x^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{k}{\omega a}) + \mathcal{O}(\frac{a}{\omega})$ neighbourhood of the origin and $\|\hat{u} - u^*\|$ enters an $\mathcal{O}(\frac{1}{\omega}) + \mathcal{O}(\frac{1}{\omega a \tau_I}) + \mathcal{O}(\frac{a}{\tau_I \omega})$ of the origin.

Proportional Integral ESC

- Proof of theorem demonstrates that:
  - the proportional action minimizes the impact of the time scale separation
  - the integral action acts as a standard perturbation based ESC
  - Combined action guarantees stabilization of the unknown equilibrium
  - With fast convergence
- Impact of dither signal is inversely proportional to the frequency
- Size of ROA is proportional to $\frac{a}{k}$.
- **PIESC acts as a dynamic output feedback nonlinear controller.**

Example 1

We consider the following dynamical system:

$$\dot{x}_1 = -x_1 + u$$

The cost function to be minimized is given by: $y = 1 + 4(x_1 - 1.2)^2$.

- the optimum cost is $y^* = -1.25$ and occurs at $x_1^* = 1.2$, $u^* = 1.2$
- The tuning parameters are chosen as: $a = 5$, $\omega = 100$, $k = 0.5$, $\tau_I = 1$ with $\omega_h = 1000$.
- Compared to standard ESC with $a = 5$, $\omega = 100$, $\omega_l = 150$, $\omega_h = 100$ and $\tau_I = 0.05$.

Example 1

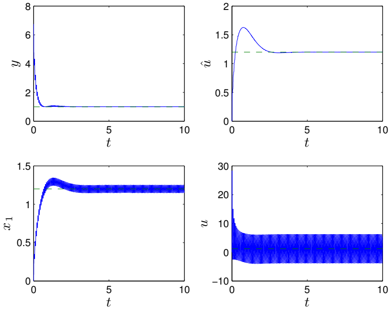


Figure: PIESC

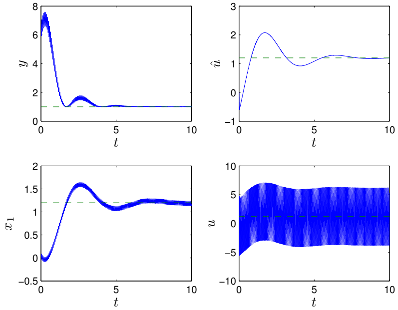


Figure: Standard ESC.

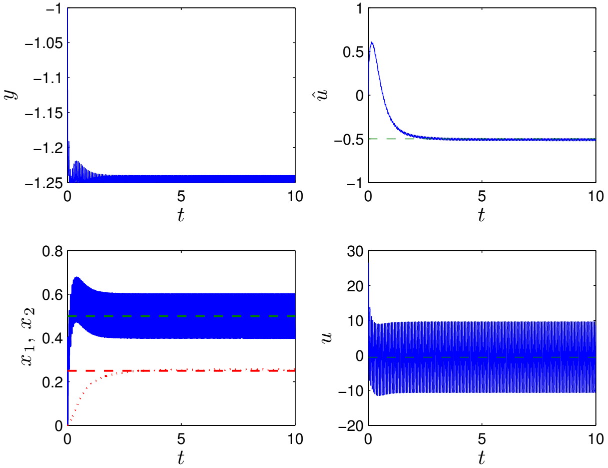
Example 2

We consider the following dynamical system taken from Guay and Zhang [2003]:

$$\dot{x}_1 = x_1^2 + x_2 + u$$
$$\dot{x}_2 = -x_2 + x_1^2$$

The cost function to be minimized is given by: $y = -1 - x_1 + x_1^2$.

- the optimum cost is $y^* = -1.25$ and occurs at $u^* = -0.5$, $x_1^* = 0.5$, $x_2^* = 0.25$
- The tuning parameters are chosen as: $k = 10$, $\tau_I = 0.1$, $a = 10$, $\omega = 100$ with $\omega_h = 1000$.
- Outperforms the model-based approach of Guay and Zhang [2003]



−1
−1.05
−1.1
−1.15
−1.2
−1.25
y
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0
−0.5
−1
û
0
5
10
t
0.8
0.6
0.4
0.2
0
x 1, x 2
0
5
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30
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−10
−20
u
0
5
10
t

Proportional Integral ESC

- Higher order output dynamics (subject to stable zero dynamics).
- Use measured derivatives of cost to synthesize a rel. order one cost.

e.g. Relative order 2 dynamics:

$$\dot{y} = L_f h,\ \ddot{y} = L_f^2 h + L_g L_f h u.$$

where $L_g L_f h \neq 0\ \forall x \in \mathcal{D}(u)/x^*$. Assume that $z_2 = \dot{y}$ is available for measurement. The cost function for PI-ESC becomes:

$$H(x) = h(x) + \frac{1}{2} z_2^2.$$

Example 3

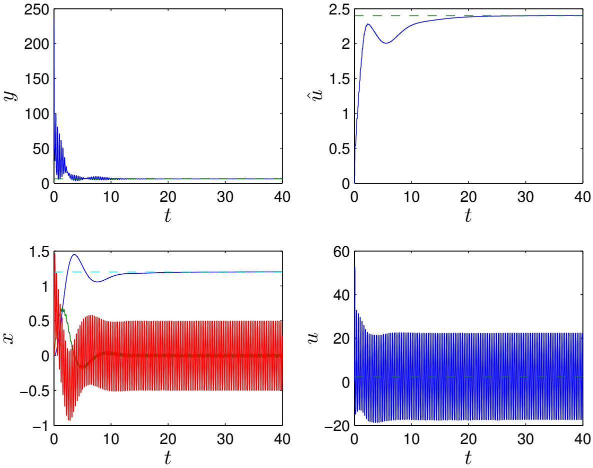
Consider the following system

$$\begin{aligned}\dot{x}_1 =& x_2\\ \dot{x}_2 =& x_3\\ \dot{x}_3 =& -x_1 - 3x_2 - 3x_3 + 0.5u_1\end{aligned}$$

with the following cost function: $y = 1 + 4(x_1 - 1.2)^2$.

- Consider the extended cost: $Y = y + (y + \dot{y})^2 + (2y + \dot{y} + \ddot{y})^2$
- Tuning parameters: $a = 20$, $\omega = 20$, $\omega_h = 100$, $k = 0.2$ and $\tau_I = 15$.
- The initial conditions are $x_1(0) = x_2(0) = x_3(0) = \hat{u}(0) = 0$.

Example 3



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2
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−0.5
−1
x
0 10 20 30 40
t
60
40
20
0
−20
u
0 10 20 30 40
t

Example 3

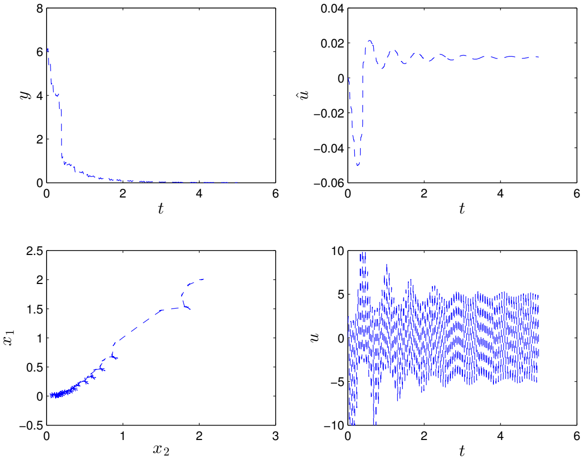
Consider the following unicycle system

$$
\begin{aligned}
\dot{x}_1 =& u\cos(x_3) \\
\dot{x}_2 =& u\sin(x_3) \\
\dot{x}_3 =& v
\end{aligned}
$$

with the following cost function: $y = \frac{1}{2}x_1^2 + x_2^2$.

- Tuning parameters: $a = 5$, $\omega = 200$, $\omega_h = 1000$, $k = 2$ and $\tau_I = 10$.
- The initial conditions are $x_1(0) = x_2(0) = 2$, $x_3(0) = 0$, $\hat{u}(0) = 0$.
- We consider a constant angular velocity $v = \sqrt{\omega}$.

Example 4



8
6
4
2
0
y
0
2
4
6
t
0.04
0.02
0
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−0.04
−0.06
$\hat{u}$
0
2
4
6
t
2.5
2
1.5
1
0.5
0
−0.5
x_1
0
1
2
3
x_2
10
5
0
−5
−10
u
0
2
4
6
t

Example 5

- Approach can be applied to time-varying RTO problems.

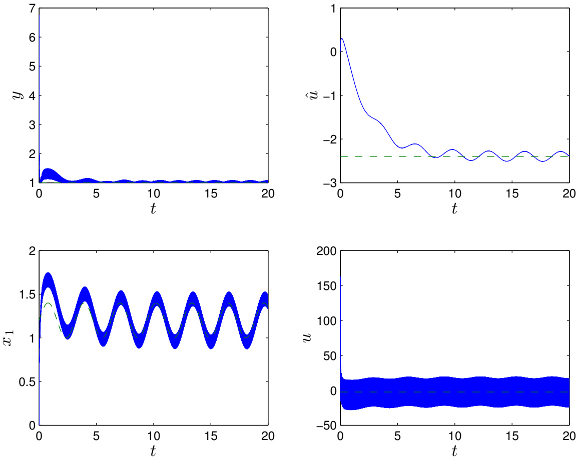
Consider the nonlinear system:

$$\dot{x}_1 = 0.5x_1 + 0.1x_1^2 + u$$

with the following cost function: $y = 1 + 4(x_1 - 1.2 - 0.2\sin(2t))^2$.

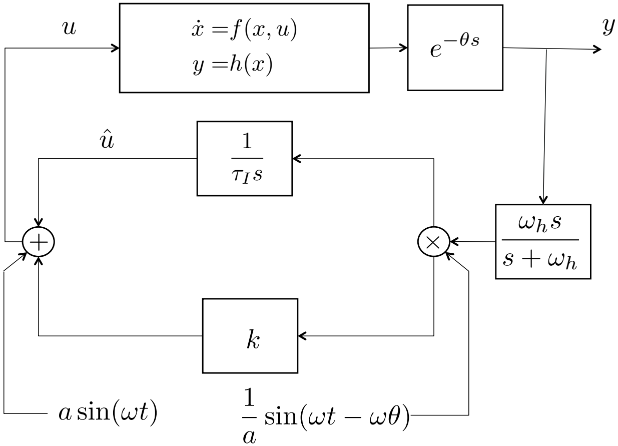
- Tuning parameters: $a = 10$, $\omega = 100$, $\omega_h = 1000$, $k = 4$ and $\tau_I = 1$.
- The initial conditions are $x_1(0) = 0$, $\hat{u}(0) = 0$.

Example 5



Proportional-Integral ESC with delay compensation

- Delay systems with known delay can be treated.
- Strategy is to compensate for phase change in dither signal



Example 6

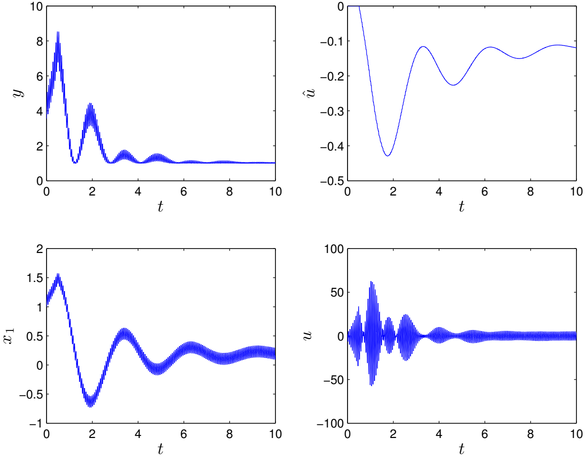
Consider the following system

$$\dot{x}_1 = 0.5x_1 + 0.1x_1^2 + u$$

with the following cost function: $y = 1 + 4(x_1(t-\theta) - 0.2)^2$.

- $\theta = 0.5$
- Tuning parameters: $A = 10$, $\omega = 100$, $\omega_h = 1000$, $k = 0.5$ and $\tau_I = 10$.
- The initial conditions are $x_1(0) = 1$, $\hat{u}(0) = 0$.

Example 6



ESC Observer design: Motivation

- The design of observers remains a challenge in nonlinear systems
- Very few design techniques exist:
  - **Differential geometric techniques:** yield design subject to restrictive assumptions
  - **Optimization based techniques:** not reliable for real-time implementation for nonlinear systems
  - **Particle filtering:** computationally inefficient due to sampling with no provable convergence
- Objective is to propose a new alternative observer design that exploits the PI extremum-seeking control approach.

PIESC for observer design

We consider a class of nonlinear systems of the form:

$$\dot{x} = f(x) \tag{4}$$

$$y = h(x) \tag{5}$$

where

- $x \in \mathbb{R}^n$ is the vector of state variables,
- $y \in \mathbb{R}$ is the output variable available for measurement.
- $f(x)$ and $h(x)$ are smooth vector valued functions of x.

Assumption

The system (4), (5) *is observable.*

PIESC observer design

- We consider the cost

$$V = \frac{1}{2}(h(x) - h(\hat{x}))^2. \tag{6}$$

- Proposed PIESC observer:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + K(t)(y - h(\hat{x})) && (7) \\ \dot{K}_i &= -\frac{1}{a\tau_I}\dot{V}D(\omega) && (8) \\ K(t) &= -\frac{k_g}{a}\dot{V}D(\omega) + aD(\omega) + K_i && (9) \end{aligned}$$

- Aims to minimize V by manipulation of the observer gain $K(t)$.
- $\dot{V}$ is estimated using a high-pass filter.

PIESC observer

Tuning parameters are:

- k_g is the proportional gain.
- τ_I is the integral constant.
- $D(\omega) = [\sin(\omega_1 t), \ldots, \sin(\omega_n t)]$
- ω_i are positive constants such that
  1. $\frac{\omega_i}{\omega_j}$ are rational,
  2. $\omega_i \neq \omega_j$ for $i \neq j$ and
  3. $\omega_k \neq \omega_i + \omega_j$

  for all i, j and $k \in [1, \ldots, n]$.

PIESC observer

Analysis proceeds in three steps:

1. Averaging of the error dynamics $e = x - \hat{x}$:

$$\begin{aligned}\dot{e} &= f(x) - f(\hat{x}) - K(t)(h(x) - h(\hat{x}))\\ \dot{K}_i &= -\frac{1}{a\tau_I}\dot{V}D(\omega)\\ K(t) &= -\frac{k_g}{a}\dot{V}D(\omega) + aD(\omega) + K_i\end{aligned}$$

2. Stability analysis of the averaged system,
3. Averaging analysis to compute deviation of the real system from the averaged system:

$$\|e_{av}(t) - e(t)\|^2 \leq \beta(a, k_g, \tau_I, D(\omega))$$

PIESC observer: LTI system analysis

- First focus on observable LTI systems
- PIESC observer:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + K(t)C(x - \hat{x}) \\ \dot{K}_i &= -\frac{1}{a\tau_I}\dot{V}D(\omega) \\ K(t) &= -\frac{k_g}{a}\dot{V}D(\omega) + aD(\omega) + K_i.\end{aligned} \tag{10}$$

where $V = \frac{1}{2}(Ce)^2$.

- Error dynamics:

$$\dot{e} = Ae - K(t)Ce \tag{11}$$

PIESC observer: LTI averaged system

- Averaged error dynamics yields:

$$\dot{e}_{av} = \Omega(A - K_i C)e_{av} - a\Gamma(Ce_{av})e_{av}$$

$$\dot{K}_{i,av} = -\frac{Ce_{av}}{a\tau_I}\Gamma(Ce_{av})(A - K_{i,av}C)e_{av} + \frac{a}{k_g\tau_I(Ce_{av})}\Gamma(Ce_{av})e_{av}$$

- For $C = [1, 0, \ldots, 0]$ (wlog), we have:

$$\Omega = \begin{bmatrix} 1 + \psi_1\left(\left(\frac{k_g}{a}\right)(Ce_{av})^2\right) & 0 & 0 & \ldots & \\ \psi_{r_1}\left(\left(\frac{k_g}{a}\right)(Ce_{av})^2\right) & 1 & 0 & \ldots & 0 \\ \psi_{r_2}\left(\left(\frac{k_g}{a}\right)(Ce_{av})^2\right) & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \psi_{r_{m-1}}\left(\left(\frac{k_g}{a}\right)(Ce_{av})^2\right) & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

and

$$\Gamma(Ce_{av}) = \frac{a}{k_g(Ce_av)^2}(\Omega - I).$$

PIESC observer: LTI averaged system

- The functions ψ_i given by:

$$\begin{aligned}
\psi_1(\sigma) &= \frac{1}{\sqrt{1-\sigma^2}} - 1 \\
\psi_3(\sigma) &= 3\left(\frac{1}{\sqrt{1-\sigma^2}} - 1\right) - \frac{4}{\sigma^2}\left(\frac{1}{\sqrt{1-\sigma^2}} - 1 - \frac{1}{2}\sigma^2\right) \\
\psi_5(\sigma) &= 5\left(\frac{1}{\sqrt{1-\sigma^2}} - 1\right) - \frac{20}{\sigma^2}\left(\frac{1}{\sqrt{1-\sigma^2}} - 1 - \frac{1}{2}\sigma^2\right) \\
&\quad + \frac{16}{\sigma^4}\left(\frac{1}{\sqrt{1-\sigma^2}} - 1 - \frac{1}{2}\sigma^2 - \frac{3}{8}\sigma^4\right) \cdots
\end{aligned}$$

  arise from the averaging of the error dynamics.
- They are related to Chebyshev polynomials of the third kind:
  - Each subscript is associated with the choice of frequencies.
  - $\psi_r(\sigma)$ are positive definite for $r = 4k+1$ for $\sigma^2 < 1$, $k = 0, 1, 2, \ldots$

PIESC observer: LTI averaged system

Resulting observer is reminiscent of Chandrasekhar-type algorithms (Kailath [1972], Lindquist [1974])

- Observer (and LQR control) without the need for the solution of Riccati equations.

Theorem

Consider the average error dynamics with average gain updates. Then there exists k_g, a, τ_I and a set of frequencies (ω_i) such that the origin is an asymptotically stable equilibrium for all $e_{av}(0)$ with $(Ce_{av}(0))^2 < 1$.

- By averaging analysis with ω_1 as a perturbation parameter. We get:

Theorem

Consider the PIESC observer error dynamics and PI gain update. Then there exists k_g, a, τ_I and a set of frequencies (ω_i) such that the estimation error dynamics enter an $\mathcal{O}(1/\omega_1)$ neighbourhood of the origin asymptotically.

Example 1

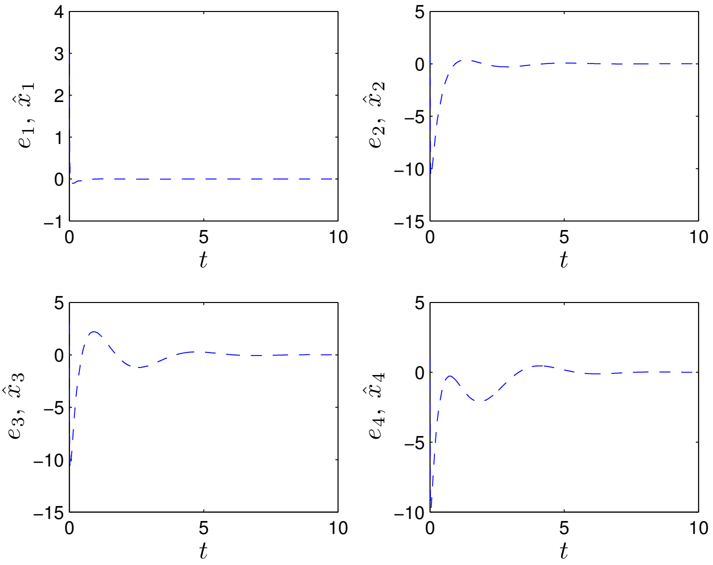
Consider the dynamical system:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + x_3,\ \dot{x}_3 = -x_2 + x_3,\ \dot{x}_4 = -x_3$$

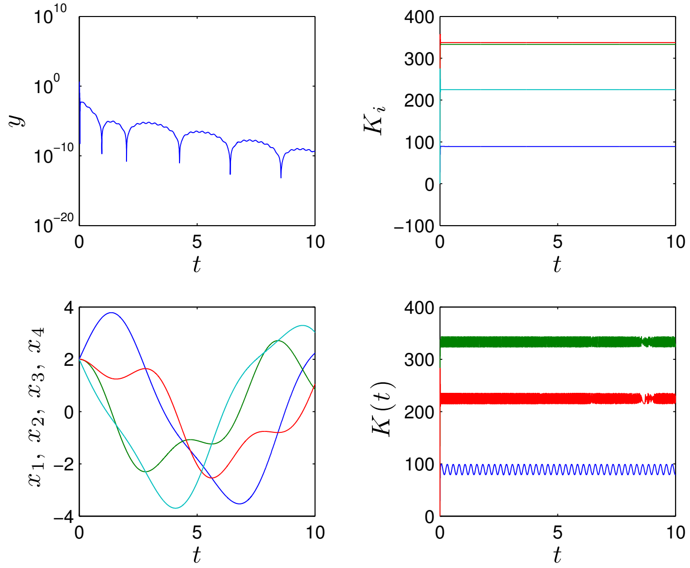
It is assumed that $y = x_1$ is available for measurement. We apply the proposed ESC observer with $A = 20$, $k_g = 10$, $\tau_I = 0.001$ and the dither signal

$$D(\omega)^\top = \begin{bmatrix} \sin(25t) & \sin(125t) & \sin(225t) & \sin(325t) \end{bmatrix}.$$

Example 1



Example 1



10^{10}
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10
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2
0
−2
−4
$x_1,\ x_2,\ x_3,\ x_4$
0
5
10
t
400
300
200
100
0
$K(t)$
0
5
10
t

Design of nonlinear observers

- Analysis of the PIESC nonlinear observer is considerably more difficult.
  - Follows the same steps as the linear observer design analysis.
  - But resulting averaged nonlinear error dynamics show that the PIESC formulation provides access to elements that are not measured: e.g. $\frac{\partial h}{\partial x}(x)$
- Approach works for systems with "averaged" Lipschitz properties (potential to handle discontinuous dynamics).
- Provides a general observed design for observable nonlinear systems.

Example 2

Consider the dynamical system (Andrieu et al. [2009]):

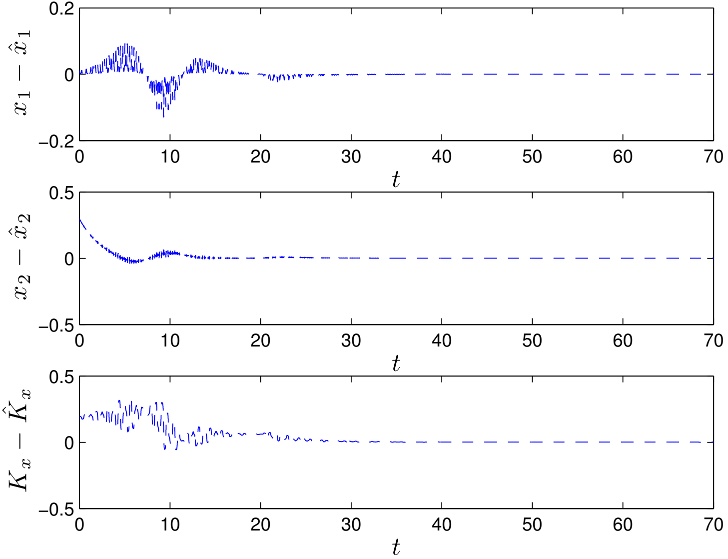
$$\dot{x}_1 = \frac{x_1 x_2}{K_x x_1 + x_2} - u x_1$$
$$\dot{x}_2 = -\frac{x_1 x_2}{K_x x_1 + x_2} + u(1 - x_2)$$

where

$$u(t) = \begin{cases} 0.41 & t < 10, \quad 0.02 \quad 10 \leq t < 20, \\ 0.6 & 20 \leq t < 35, \quad 0.1 \quad \geq 35. \end{cases}$$

- Objective is to estimate x_2 and K_x using $y = x_1$.
- **Problem not solvable using high-gain observer techniques**
- Use frequencies $\omega_1 = 25$, $\omega_2 = 125$, $\omega_3 = 5$.
- Tuning parameters are $k_g = 1$, $\tau_I = 0.01$, $a = 30$.

Example 2



0.2
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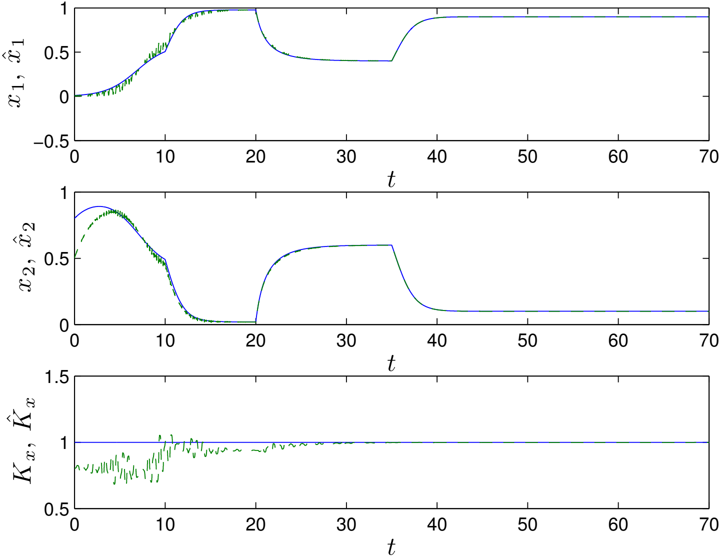
−0.2
$x_1 - \hat{x}_1$
0 10 20 30 40 50 60 70
t
0.5
0
−0.5
$x_2 - \hat{x}_2$
0 10 20 30 40 50 60 70
t
0.5
0
−0.5
$K_x - \hat{K}_x$
0 10 20 30 40 50 60 70
t

Example 2


$x_1,\ \hat{x}_1$
1
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0
−0.5
0 10 20 30 40 50 60 70
t
$x_2,\ \hat{x}_2$
1
0.5
0
0 10 20 30 40 50 60 70
t
$K_x,\ \hat{K}_x$
1.5
1
0.5
0 10 20 30 40 50 60 70
t

Example 3

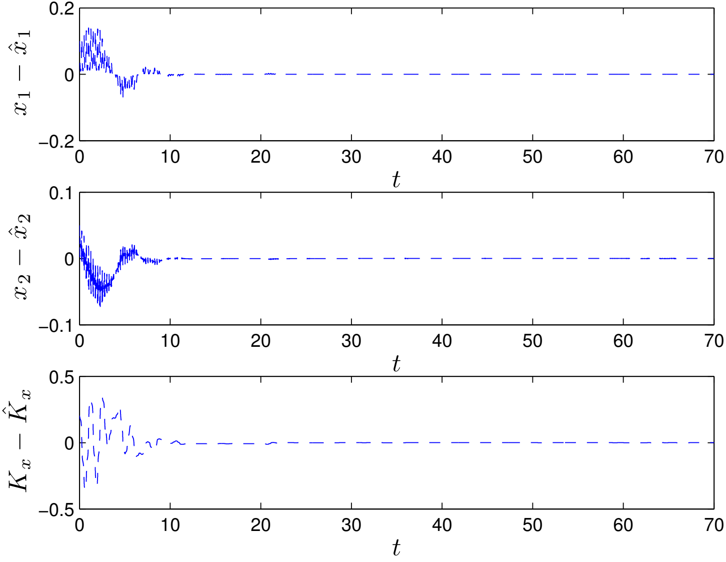
Consider the bioreactor system with Monod kinetics:

$$\begin{aligned}\dot{x}_1 &= \frac{x_1 x_2}{K_s + x_2} - u x_1 \\ \dot{x}_2 &= -\frac{x_1 x_2}{K_s + x_2} + u(1 - x_2)\end{aligned}$$

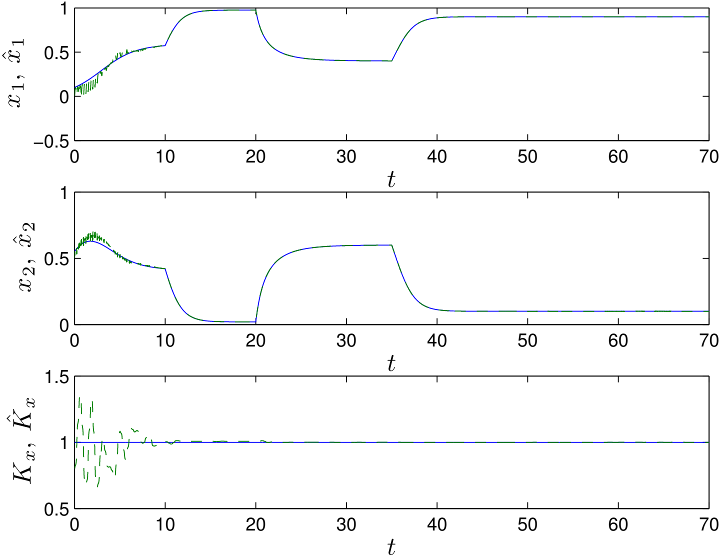
where

$$u(t) = \begin{cases} 0.41 & t < 10, \quad 0.02 \quad 10 \le t < 20, \\ 0.6 & 20 \le t < 35, \quad 0.1 \quad \ge 35. \end{cases}$$

- Objective is to estimate x_1 and K_s using $y = x_2$.
- **Problem requires slow (asymptotic) observers**
- Use frequencies $\omega_1 = 25$, $\omega_2 = 125$, $\omega_3 = 5$.
- Tuning parameters are $k_g = 1$, $\tau_I = 0.01$, $a = 30$.


0.2
0
−0.2
$x_1 - \hat{x}_1$
0 10 20 30 40 50 60 70
t
0.1
0
−0.1
$x_2 - \hat{x}_2$
0 10 20 30 40 50 60 70
t
0.5
0
−0.5
$K_x - \hat{K}_x$
0 10 20 30 40 50 60 70
t

Example 3


$x_1,\ \hat{x}_1$
1
0.5
0
−0.5
0
10
20
30
40
50
60
70
t
$x_2,\ \hat{x}_2$
1
0.5
0
0
10
20
30
40
50
60
70
t
$K_x,\ \hat{K}_x$
1.5
1
0.5
0
10
20
30
40
50
60
70
t

Conclusions

- A proportional-integral ESC structure is developed to eliminate time-scale separation
  - Proportional action provides quick transient response
  - Integral action computes the correct optimal steady-state
- Shown to stabilize a class of minimum phase nonlinear systems to a neighbourhood of the unknown optimum.
- Proposed a systematic PI ESC observer approach was a large class of detectable nonlinear systems.

Future and Ongoing Work:

- Study extension to nonminimum phase cost dynamics and unstable dynamics
- Generalize to discrete-time dynamics
- Application in distributed optimization over networks of dynamic local agents.

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